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Hydromagnetic wavelike instabilities in a rapidly rotating stratified fluid

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We examine the hydromagnetic stability of a radially stratified fluid rotating between two coaxial cylinders, with particular emphasis on the case when the angular velocity greatly exceeds both buoyant and Alfvén frequencies. If the magnetic field is predominantly azimuthal instabilities then have an essentially non-axisymmetric and wavelike character. Various bounds on their phase speeds and growth rates are derived, including a 'quadrant' theorem analogous to Howard's semicircle theorem for Kelvin–Helmholtz instability. Their strong tendency to propagate *against* the basic rotation (i.e. 'westward'), previously noted by the author in the study of a more simplified (homogeneous) model, seems relatively insensitive to the generation mechanism (e.g. unstable gradient of magnetic field, angular velocity or density), but a number of counterexamples show that this constraint need not apply if the magnetic field displays significant spatial variations of direction as well as magnitude and that eastward-propagating amplifying modes are then possible.

1. Introduction

In a recent paper (Acheson 1972, hereafter referred to as A) the author has examined a class of hydromagnetic instabilities in a uniformly rotating homogeneous incompressible fluid that arise from variations of the (azimuthal) magnetic field with distance from the rotation axis. At low values of the parameter $\mathscr{L} \equiv V/\Omega R$ these assume the form of non-axisymmetric 'slow' hydromagnetic waves, with both frequency and growth rate typically of order $V^2/\Omega R^2$ (see, for example, Acheson & Hide 1973), and propagate *against* the basic rotation. Here V denotes a typical Alfvén speed, Ω the angular velocity of rotation and R a dimension of the system. The possibility that within the earth's liquid core such waves may be responsible, in part at least, for the slow westward drift with time of the geomagnetic field (see, for example, Hide & Stewartson 1972) gains support from the above constraint on their azimuthal propagation, and in attempting here to take some account of additional effects due to buoyancy and differential rotation we try to bridge the gap between the highly idealized system of A and current ideas concerning the dynamically important agencies within the core (see, for example, Roberts & Soward 1972). It proves expedient from a mathematical viewpoint to take, as in A, cylindrical (rather than spherical,

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as a strict regard for the geophysical problem would require) boundaries for our simplified model. While the whole question of how sensitive the results are to the boundary conditions is as yet unresolved, a few remarks on the matter are made in § 7.

The basic stability problem is formulated mathematically in § 2 (see equation (2.6) for the equilibrium configuration). An axial shear flow $U_z(r)$ is carried in this formulation with little extra expense in terms of algebraic manipulation, although its effects are investigated only briefly in the appendix. An elementary extension of some results by Howard & Gupta (1962) is presented there and a non-hydromagnetic analogue of one of the main themes of both this paper and A, namely that (when rapidly rotating) the system is far more unstable to non-axisymmetric than symmetric disturbances, is noted. While an investigation of axisymmetric instability (§ 3) is therefore a little academic at rapid rotation speeds, the physical interpretation following the instability criterion (3.5) provides a clear picture of why non-axisymmetric modes are so much more readily excited. In § 4 it is shown that in a uniformly rotating fluid permeated by an azimuthal magnetic field all amplifying waves propagate westward, whether generated by an unstable gradient of magnetic field, density or both.† In a *non*-uniformly rotating fluid the constraint on their azimuthal propagation is naturally rather less stringent, but amplifying waves nevertheless always propagate westward *relative to the fastest rotating portion of the fluid*, despite the fact that they may then be deriving their energy from any one (or more) of *three* quite different sources. General conditions necessary for the amplification of non-axisymmetric modes are established in § 5 (see equation (5.5)). In the latter parts of both §§ 4 and 5 attention is focused on the 'slow' waves characteristic of a rapidly rotating fluid and elementary bounds on their phase speeds and growth rates (see (4.9) and (5.9)) are derived. In § 6 the general results of the preceding sections are illustrated by two specific examples, which show that the sense of azimuthal propagation is not *entirely* insensitive to the magnetic field configuration and that there are circumstances in which amplifying waves may propagate towards the east.

2. Mathematical formulation

When all transport processes (i.e. viscosity, electrical resistance, thermal diffusion etc.) can be neglected the basic hydromagnetic equations governing the motion of an incompressible fluid are

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\,.\,\nabla\mathbf{u}\right) = -\nabla p+\frac{1}{\mu}\,\mathbf{B}\,.\,\nabla\mathbf{B}+\rho\mathbf{g}, \tag{2.1}$$

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{2.2}$$

$$\nabla\,.\,\mathbf{u} = 0, \quad \nabla\,.\,\mathbf{B} = 0, \qquad (2.3), (2.4)$$

$$\partial\rho/\partial t+\mathbf{u}\,.\,\nabla\rho = 0 \tag{2.5}$$

(see, for example, Chandrasekhar 1961). Here ρ denotes the local fluid density, $\mathbf{u}$ the Eulerian velocity vector, t time, μ magnetic permeability, $\mathbf{B}$ magnetic

† See 'Note added in proof', part (i), p. 623.

field, $\mathbf{g}$ acceleration due to gravity and $p = p_F + \frac{1}{2}\mu^{-1}\mathbf{B}^2$ is the 'total' pressure, including both the actual fluid pressure p_F and the magnetic pressure $\frac{1}{2}\mu^{-1}\mathbf{B}^2$.

Referring all quantities to a set of cylindrical polar co-ordinates (r, θ, z), the basic equilibrium state

$$\mathbf{u}_0 = \{0, U_\theta(r), U_z(r)\}, \quad \mathbf{B}_0 = \{0, B_\theta(r), B_z(r)\}, \quad \rho_0 = \rho_0(r), \tag{2.6}$$

representing a spiralling flow in the presence of radial gradients of both magnetic field and density, is an exact solution of (2.1)–(2.5) provided that the gravitational body force $\mathbf{g} = \{g(r), 0, 0\}$ per unit mass is purely radial. The gravitational and centrifugal forces, together with the radial component of the 'tension' in the 'equivalent elastic strings' associated with the curvature of the magnetic field lines are exactly balanced by a radial pressure gradient

$$\frac{dp_0}{dr} = \rho_0\left(g + \frac{U_\theta^2}{r}\right) - \frac{B_\theta^2}{\mu r}. \tag{2.7}$$

We here consider the stability of such a flow between two coaxial cylinders of infinite length, radii r_1 and r_2.

Thus if we slightly disturb the system the linearized forms of (2.1)–(2.5) admit solutions in which (by virtue of the equilibrium configuration (2.6)) all perturbation quantities ϕ may be written as

$$\phi = \mathscr{R}[\hat{\phi}(r)\exp i(m\theta + nz - \sigma t)], \tag{2.8}$$

where m, n and σ are constants. The last of these (which may be complex) represents the frequency of oscillation as seen by an inertial observer, and it will prove convenient to define a Doppler-shifted frequency

$$\omega(r) \equiv \sigma - mU_\theta/r - nU_z, \tag{2.9}$$

which is that measured by an observer rotating with the local angular velocity of the fluid and moving in the z direction with the local axial flow.

Subject to the Boussinesq approximation, in which the basic gradient of density is supposed so weak that ρ_0 may be treated as constant and replaced by its mean value $\bar{\rho}_0$ everywhere in (2.1) save in the buoyancy term, we eliminate all perturbation variables in favour of the radial velocity component $\hat{u}_r(r)$. Thus, defining the local Alfvén speeds

$$V_\theta(r) \equiv B_\theta(r)/(\mu\bar{\rho}_0)^{\frac{1}{2}}, \quad V_z(r) \equiv B_z(r)/(\mu\bar{\rho}_0)^{\frac{1}{2}}, \tag{2.10}$$

the Brunt–Väisälä frequency

$$N(r) \equiv \left[\frac{\rho_0'}{\rho_0}\left(g + \frac{U_\theta^2}{r}\right)\right]^{\frac{1}{2}} \tag{2.11}$$

(which is real if the density distribution is 'bottom-heavy' but purely imaginary otherwise) and the functions

$$F(r) \equiv (mV_\theta/r + nV_z)^2 - \omega^2, \tag{2.12}$$

$$Q(r) \equiv 2\left[\frac{U_\theta}{r}\omega + \frac{V_\theta}{r}\left(\frac{mV_\theta}{r} + nV_z\right)\right], \tag{2.13}$$

$$G(r) \equiv F + r\left(\frac{U_\theta^2}{r^2}\right)' - r\left(\frac{V_\theta^2}{r^2}\right)' + N^2 - \frac{Q^2}{F}, \tag{2.14}$$

all of which have dimension (frequency)², we find

$$F\psi'' + \left\{F' + \frac{F}{r}\left(\frac{r^2+3m^2n^{-2}}{r^2+m^2n^{-2}}\right)\right\}\psi' + H\psi = 0, \tag{2.15a}$$

where

$$H(r) \equiv \frac{2n^2V_zV_z'}{r} + n^2r\left(\frac{V_\theta^2}{r^2}\right)' - n^2r\left(\frac{U_\theta^2}{r^2}\right)' - N^2\left(\frac{m^2}{r^2}+n^2\right) + \frac{Q^2n^2}{F}$$
$$+\frac{2mQ}{r^2+m^2n^{-2}} - \frac{F}{r^2+m^2n^{-2}}\left\{r^2n^2+1+2m^2+\frac{m^2}{n^2r^2}(m^2-1)\right\} + \frac{2nU_z'}{r}(\sigma - nU_z). \tag{2.15b}$$

Here $\psi(r) \equiv i\hat{u}_r/\omega$ and primes denote differentiation with respect to r. In view of the boundary conditions solutions of (2.15) are subject to $\psi(r_1) = \psi(r_2) = 0$. We emphasize here that (except in the appendix) we take $U_z = 0$, with two simple consequences for the above equations: the final terms in (2.9) and (2.15b) vanish.

In the two examples of §6 it proves more helpful to formulate the problem in terms of the perturbations $\hat{p}$ to the total pressure, in which case

$$\hat{p}'' + \left(\frac{1}{r} - \frac{G'}{G}\right)\hat{p}' + \left\{\frac{m}{r}\left(\frac{Q}{F}\right)' - \frac{mQG'}{rFG} - \frac{G}{F}\left(\frac{m^2}{r^2}+n^2\right) - \frac{m^2Q^2}{r^2F^2}\right\}\hat{p} = 0 \tag{2.16a}$$

takes the place of (2.15a) and is subject to the boundary conditions

$$\hat{p}' + (mQ/rF)\hat{p} = 0 \quad \text{at} \quad r = r_1, r_2. \tag{2.16b}$$

3. Stability with respect to axisymmetric disturbances

We confine attention here to the case in which the magnetic field is purely azimuthal, i.e. $V_z = 0$. When $m = 0$ equation (2.15a) reduces to

$$\psi'' + \frac{1}{r}\psi' - \left(\frac{n^2L}{\omega^2} + n^2 + \frac{1}{r^2}\right)\psi = 0, \tag{3.1}$$

where

$$L(r) \equiv r\frac{d}{dr}\left(\frac{V_\theta^2}{r^2}\right) - N^2 - \frac{1}{r^3}\frac{d}{dr}(rU_\theta)^2, \tag{3.2}$$

and ω becomes identical with the (constant) frequency σ measured by an inertial observer (see equation (2.9)). Introducing the transformation $\xi = r^{\frac{1}{2}}\psi$ we find

$$\xi'' - \left(\frac{n^2L}{\omega^2} + n^2 + \frac{3}{4r^2}\right)\xi = 0. \tag{3.3}$$

On multiplying (3.3) by the complex conjugate of ξ and integrating over the interval $r_1 \leqslant r \leqslant r_2$ (making use of the boundary conditions $\xi(r_1) = \xi(r_2) = 0$) we have

$$\omega^2\int_{r_1}^{r_2}\left\{|\xi'|^2 + \left(n^2+\frac{3}{4r^2}\right)|\xi|^2\right\}dr = -\int_{r_1}^{r_2} Ln^2|\xi|^2\,dr, \tag{3.4}$$

from which it is clear that ω^2 is real and that disturbances therefore either oscillate about the equilibrium position without amplifying ($\omega^2 > 0$) or grow aperiodically ($\omega^2 < 0$). Thus if $L(r)$ has the same sign throughout the interval $r_1 \leqslant r \leqslant r_2$, ω^2 has the opposite sign and the system will be stable or unstable according as $L \leqslant 0$ or $L > 0$. Further, if L changes sign, it has been established in Sturm–Liouville theory, which applies here, that both positive and negative values of ω^2 will occur (see, for example, Ince 1944, p. 235). The system is therefore stable to axisymmetric disturbances *if and only if* $L \leqslant 0$ everywhere in the fluid, and this result is a straightforward extension of that obtained by Michael (1954) to include effects due to radial density stratification (see also Roberts & Soward 1972). It is evident from (3.2) that a radial decrease of angular momentum is destabilizing (Rayleigh 1920), as is a 'top-heavy' density gradient ($N^2 < 0$) or a radial *increase* of V_θ^2/r^2. By inspection of (3.4) we in fact have the following simple bound on the growth rates of any amplifying disturbances: $\omega_I^2 < \max L$ ($\omega \equiv \omega_R + i\omega_I$).

A case of especial interest is that of uniform rotation with angular velocity Ω. The system will then be stable unless the magnetic field and density distributions are such that

$$r(V_\theta^2/r^2)' - N^2 > 4\Omega^2 \tag{3.5}$$

somewhere in the fluid. Clearly if the fluid rotates 'rapidly', in the sense that

$$V_*^2/r_*^2 + N_*^2 \ll \Omega^2 \tag{3.6}$$

(where an asterisk denotes 'typical magnitude of'), and the magnetic field gradient is moderate (e.g. $B_\theta \propto r^3$) the system is then thoroughly stable to axisymmetric disturbances.

An investigation of the energetics behind (3.5) proves helpful in the physical interpretation of the results in §§5 and 6. Consider the interchange of two thin rings of fluid, both of volume τ, one initially situated at $r = r_a$ permeated by an azimuthal magnetic field B_a and the other initially at $r = r_b$ permeated by a magnetic field B_b. Denote their angular velocities and densities similarly by U_a/r_a and U_b/r_b, and ρ_a and ρ_b respectively. We investigate first the increase (or decrease) in magnetic energy resulting from such an exchange. If Δ is the cross-sectional area of any ring the magnetic flux threading it is $B\Delta$, which must remain constant by virtue of the perfect conductivity of the fluid. Since the fluid is incompressible the ring's volume $2\pi r\Delta$ must also remain constant, so the quantity B/r is conserved for each ring during its motion. The magnetic fields permeating the rings in their new positions are thus $B_b r_a/r_b$ and $B_a r_b/r_a$. The concomitant increase in magnetic energy is therefore

$$-\tau(B_b^2/r_b^2 - B_a^2/r_a^2)(r_b^2 - r_a^2)/2\mu$$

and unless B_θ^2/r^2 is everywhere a decreasing function of r it is possible to select rings for which this energy change is negative. In this way the radial increase of an azimuthal magnetic field can lead to instability.

This tendency may or may not be promoted by the basic radial density gradient, the increase in gravitational potential energy due to the ring exchange being

$$\tau(\rho_b - \rho_a)\int_{r_a}^{r_b} (g + U_\theta^2/r)\, dr.$$

Further, we have yet to consider the concomitant change of kinetic energy. In this respect we note that since axisymmetric disturbances to the azimuthal magnetic field are themselves azimuthal the induced hydromagnetic torque $\nabla \times (\mathbf{B}.\nabla\mathbf{B})$ on individual fluid elements vanishes everywhere. Further, the axial component of the buoyancy torque $(\nabla\rho) \times \mathbf{g}$ is zero when the perturbations are axisymmetric and $\mathbf{g}$ has no azimuthal component. There is therefore no axial torque on any fluid ring, which must accordingly conserve its angular momentum as it expands or contracts. The velocities of the rings in their new positions are thus $U_b r_b/r_a$ and $U_a r_a/r_b$, so that the increase in kinetic energy resulting from the interchange is $\frac{1}{2}\bar{\rho}\tau(U_b^2 r_b^2 - U_a^2 r_a^2)(r_b^2 - r_a^2)/r_a^2 r_b^2$. If we finally let $r_b - r_a \to 0$ we find that the condition for instability $L > 0$ is equivalent to the requirement that the algebraic sum of the three changes in energy computed above should represent a net *decrease* in energy of the system.

The extremely steep increase of magnetic field with radius required for instability when the fluid is in 'rapid' uniform rotation is evidently a consequence of the large amount of work needed in this case to exchange two rings *while conserving the angular momentum of each during transit*. Such an interchange will not be subject to this constraint when performed in a *non*-axisymmetric manner, for the net axial hydromagnetic torque on an individual ring (which will, of course, be distorted during transit) will then no longer vanish. This is reflected by the results of §§5 and 6, for comparatively modest magnetic field gradients then suffice for instability. In contrast to the axisymmetric case such instability then occurs in the form of slow hydromagnetic *waves* which grow in amplitude with time, and in the next section we derive a simple constraint on their azimuthal propagation.

4. Azimuthal propagation of non-axisymmetric amplifying waves

Equation (2.15*a*) may be rewritten in the form

$$\left(\frac{r^3 F\psi'}{r^2+m^2n^{-2}}\right)' + \frac{r^3 H\psi}{r^2+m^2n^{-2}} = 0, \tag{4.1}$$

whence multiplying by the complex conjugate of ψ and integrating between $r = r_1$ and $r = r_2$ (where ψ vanishes) we find

$$\int_{r_1}^{r_2} \frac{r^3}{r^2+m^2n^{-2}} \{H(r)|\psi|^2 - F(r)|\psi'|^2\}\, dr = 0. \tag{4.2}$$

We now set $\omega = \omega_R + i\omega_I$, noting that $\omega_R = \sigma_R - mU_\theta/r$ is in general a function of r while $\omega_I = \sigma_I$ is a constant (see equation (2.9)). Taking the imaginary part of (4.2) we conclude that

$$\omega_I \int_{r_1}^{r_2} \frac{r^3}{r^2+m^2n^{-2}} \left\{\omega_R|\psi'|^2 + \left[S_1(r) + \frac{2mU_\theta}{(r^2+m^2n^{-2})\,r} + \frac{4n^2 S_2(r)}{|F|^2}\right]|\psi|^2\right\} dr = 0, \tag{4.3}$$

where
$$S_1(r) \equiv \frac{\omega_R}{r^2+m^2n^{-2}}\left\{r^2n^2 + 1 + 2m^2 + \frac{m^2}{n^2r^2}(m^2-1)\right\} \tag{4.4}$$

and $$S_2(r) \equiv \omega_R\left(\frac{mV_\theta}{r}+nV_z\right)^2\left(\frac{V_\theta^2+U_\theta^2}{r^2}\right) + \frac{U_\theta V_\theta}{r^2}\left(\frac{mV_\theta}{r}+nV_z\right)\left\{\left(\frac{mV_\theta}{r}+nV_z\right)^2+\omega_R^2+\omega_I^2\right\}. \tag{4.5}$$

We shall concentrate on the case in which the angular velocity $\Omega(r) = U_\theta/r$ of the fluid is in the same sense (positive, say) throughout the interval

$$r_1 \leqslant r \leqslant r_2,$$

and denote the (constant) angular propagation velocity σ_R/m of the disturbance relative to an inertial frame by Ω_p. By inspection of (4.3)–(4.5) it is then clear that if the magnetic field is purely azimuthal all amplifying non-axisymmetric modes in this case have $\omega_R/m < 0$ somewhere and thus

$$\Omega_p < \max\{\Omega(r)\}. \tag{4.6}$$

For a uniformly rotating homogeneous fluid this result reduces to that obtained in A, namely that all such modes propagate 'westward'. Effects due to radial density stratification evidently leave the result unchanged, even in situations when a 'top-heavy' density gradient constitutes the very source of the unstable modes under discussion.† Further, by inspection of (4.3)–(4.5) we see that even when the magnetic field has an axial component the constraint (4.6) still applies to all those modes for which $|nV_z| < |mV_\theta/r|$ throughout the interval $r_1 \leqslant r \leqslant r_2$.

Slow amplifying waves: bounds on their azimuthal phase speeds

We suppose here that (*a*) the rotation of the fluid is uniform, (*b*) the magnetic field is purely azimuthal and (*c*) radial, axial and azimuthal wavelengths are all of the same order r_* (the magnitude of the first of these being crudely defined locally as $2\pi|\psi|/|\psi'|$). Anticipating that when $\Omega^2 \gg V_*^2/r_*^2+N_*^2$ the slow modes will have $|\omega|^2 \sim V_*^2(V_*^2 r_*^{-2}+N_*^2)/\Omega^2 r_*^2$ (see §6 and Acheson & Hide 1973) we can write $F \doteqdot m^2V_\theta^2/r^2$ with error $O\{(V_*^2 r_*^{-2}+N_*^2)/\Omega^2\}$ and thus find that (4.3) may be written as

$$\Omega\omega_I\int_{r_1}^{r_2}\frac{r^5S_3(r)\,|\psi|^2}{(r^2+m^2n^{-2})\,m^2V_\theta^2}\,dr = 0 \tag{4.7}$$

with the same small degree of error, where

$$S_3(r) \equiv \omega_R m\Omega + \frac{m^2V_\theta^2}{r^2}\left(\frac{r^2+\frac{3}{2}m^2n^{-2}}{r^2+m^2n^{-2}}\right). \tag{4.8}$$

The fact that $S_3(r)$ must evidently vanish somewhere in the interval $r_1 \leqslant r \leqslant r_2$ immediately imposes the following upper and lower bounds on the azimuthal propagation speeds of the slow amplifying hydromagnetic waves:

$$\min\left\{\frac{V_\theta^2}{|\Omega|r^2}\right\} < \left|\frac{\omega_R}{m}\right| < \tfrac{3}{2}\max\left\{\frac{V_\theta^2}{|\Omega|r^2}\right\}, \tag{4.9}$$

in addition to the 'westward' propagation $\omega_R\Omega m < 0$ already proved above. We note that these particular bounds depend on neither the sign nor magnitude

† See 'Note added in proof', part (i), p. 623.

of N^2. The role of density stratification becomes, however, more evident when we derive further bounds in the next section on not only the phase speeds of these waves but also their growth rates.

5. Conditions necessary for the amplification of non-axisymmetric disturbances

We confine attention, as in the previous subsection, to the case of uniform rotation and azimuthal magnetic field, so that ω_R is constant. On dividing (4.2) by ω^2 and equating the imaginary part of the resulting left-hand side to zero we find

$$\omega_R\omega_I\int_{r_1}^{r_2}\frac{r^3}{r^2+m^2n^{-2}}\left[\frac{m^2V_\theta^2}{r^2}\left|\frac{\psi'}{\omega}\right|^2+\left\{\frac{4n^2\Omega^2|\omega|^2}{|F|^2}+S_4+S_5+S_6\right\}|\psi|^2\right]dr=0, \quad (5.1)$$

where

$$S_4(r)\equiv-\frac{n^2}{|\omega|^2}\left\{r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\left(1+\frac{m^2}{r^2n^2}\right)\right\}, \quad (5.2)$$

$$S_5(r)\equiv-\frac{2\Omega m}{\omega_R}\left[\frac{2n^2V_\theta^2}{|F|^2r^2}\left\{\frac{m^2V_\theta^2}{r^2}-3\omega_R^2+\omega_I^2\right\}+\frac{1}{r^2+m^2n^{-2}}\right], \quad (5.3)$$

$$S_6(r)\equiv\frac{m^2V_\theta^2}{r^2|\omega|^2}\left[\frac{S_1(r)}{\omega_R}-\frac{4}{r^2+m^2n^{-2}}-\frac{4n^2}{m^2}\left\{1+\frac{r^2|\omega|^4}{m^2V_\theta^2}\left(\frac{m^2V_\theta^2}{r^2}-2\omega_R^2+2\omega_I^2\right)^{-1}\right\}^{-1}\right]. \quad (5.4)$$

We have shown that any non-axisymmetric unstable mode must have

$$\omega_R m\Omega<0,$$

and if $m^2V_\theta^2\geqslant 3r^2\omega_R^2$ everywhere in the interval $r_1\leqslant r\leqslant r_2$ then $S_5(r)$ must therefore be positive. The term $S_6(r)$ can easily be shown to exceed

$$n^2(m^2-4)\,V_\theta^2/r^2|\omega|^2$$

if $|m|>1$ and $-5n^2V_\theta^2/r^2|\omega|^2$ if $|m|=1$. We thus conclude that for the amplification of modes such that $m^2V_\theta^2\geqslant 3r^2\omega_R^2$ everywhere (amongst which, for example, are the 'slow' hydromagnetic waves) the magnetic field and density distributions must be such that

$$r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\left(1+\frac{m^2}{r^2n^2}\right)>\begin{cases}(m^2-4)\,V_\theta^2/r^2 & \text{if } |m|>1,\\ -5V_\theta^2/r^2 & \text{if } |m|=1,\end{cases} \quad (5.5)$$

somewhere in the fluid.

Slow amplifying waves: a 'quadrant' theorem

As in the second part of §4 we now focus attention on the 'slow' waves characteristic of a 'rapidly' rotating fluid and suppose† that radial, axial and azimuthal wavelengths are all of the same order r_*. Anticipating again that

$$|\omega|^2\sim V_*^2(V_*^2r_*^{-2}+N_*^2)/\Omega^2r_*^2$$

we find that the coefficient of $|\psi|^2$ in (5.1) may be replaced with error

$$O\{(V_*^2r_*^{-2}+N_*^2)^2r_*^2/\Omega^2V_*^2\}$$

† See 'Note added in proof', part (ii), p. 624.

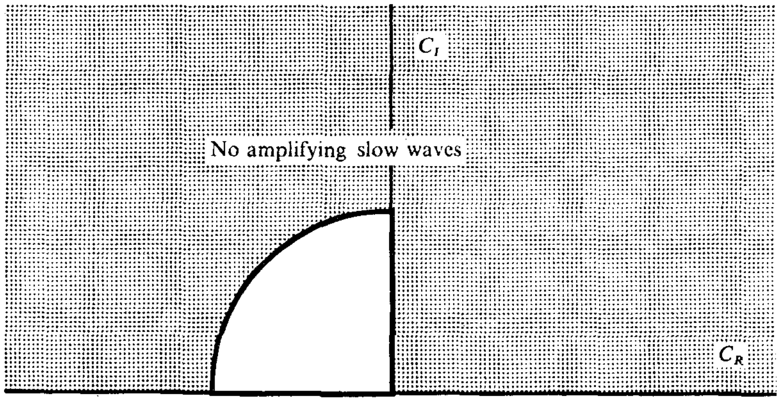

FIGURE 1. Illustrating the quadrant of the complex c plane to which slow amplifying waves are confined when the rotation is uniform and the magnetic field is azimuthal. Both m and Ω are here reckoned positive. The radius of the quadrant is $\frac{1}{4}|\Omega|^{-1}\max\{r(V_\theta^2/r^2)'-N^2\}$ when $|m|>1$ and the density distribution is 'bottom-heavy'.

by $|\omega|^{-2}S_7(r)$, where

$$S_7(r) \equiv -n^2\left\{r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\left(1+\frac{m^2}{r^2n^2}\right)\right\}-\frac{2\Omega n^2|\omega|^2}{m\omega_R}\left(\frac{2r^2+3m^2n^{-2}}{r^2+m^2n^{-2}}\right)$$
$$+\frac{m^2V_\theta^2}{r^2}\left\{\left(\frac{1}{r^2}+\frac{n^2}{m^2}\right)(m^2-4)+\left(\frac{1+3m^2/n^2r^2}{r^2+m^2n^{-2}}\right)\right\}. \quad (5.6)$$

We thus conclude that to avoid violation of (5.1) S_7 must somewhere be negative. For unstable modes with $|m|>1$ this therefore implies that (since $\Omega m\omega_R<0$; see §4)

$$4\left|\frac{\Omega}{m\omega_R}\right|(\omega_R^2+\omega_I^2) < \max\left\{r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\left(1+\frac{m^2}{r^2n^2}\right)-(m^2-4)\frac{V_\theta^2}{r^2}\right\}, \quad (5.7)$$

which in turn implies that

$$|c| < \frac{1}{4|\Omega|}\max\left\{r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\left(1+\frac{m^2}{r^2n^2}\right)-(m^2-4)\frac{V_\theta^2}{r^2}\right\}, \quad (5.8)$$

where $c\equiv\omega/m$. Thus in the presence of a 'bottom-heavy' density distribution

$$|c| < \frac{1}{4|\Omega|}\max\left\{r\left(\frac{V_\theta^2}{r^2}\right)'-N^2\right\}. \quad (5.9)$$

While for any particular mode more stringent bounds can be found by applying (5.8) this last inequality is in some respects the most useful, since the right-hand side is independent of both m and n. The bounds for the $|m|=1$ mode seem rather less stringent; the term $-(m^2-4)V_\theta^2/r^2$ in (5.8) is replaced by $5V_\theta^2/r^2$ (as indeed a comparison of (5.8) and (5.5) would suggest) and an additional term $5V_\theta^2/r^2$ accordingly appears in the curly brackets in (5.9).

Taking any of these results in conjunction with the constraint $c_R\Omega<0$ already derived in §4 we find that *the complex wave speed c of any unstable slow mode must lie within one quadrant of the complex c plane* (cf. Howard's (1961) semicircle theorem for Kelvin–Helmholtz instability), as shown in figure 1.

6. Two specific examples

We now illustrate the general results of §§3, 4 and 5 by means of two specific examples. We restrict attention in each to the case of uniform rotation.

Both B_θ/r and B_z constant, no buoyancy

In this case it proves convenient to treat the eigenvalue problem formulated in terms of the perturbation pressure. The quantities F, Q and G are all constant and (2.16a) reduces to

$$\hat{p}'' + \frac{1}{r}\hat{p}' + \left\{n^2\left(\frac{Q^2}{F^2} - 1\right) - \frac{m^2}{r^2}\right\}\hat{p} = 0. \tag{6.1}$$

We observe by inspection of (6.1) and the boundary conditions (2.16b) that Q/F, while dependent on m and n, must evidently be independent of both the magnetic field and Ω in this case. In particular, regardless of the values of B_θ/r and B_z, the quantity Q/F must retain its *non*-hydromagnetic value corresponding to the same m and n, namely $-2\Omega/\omega_0$. Here ω_0 denotes the appropriate eigenvalue of the non-hydromagnetic problem (discussed at length in Chandrasekhar 1961, §68) and will typically be of order Ω. Thus to every eigenvalue ω_0 (which is, of course, real) of the corresponding problem with B_θ and B_z set equal to zero there correspond two eigenvalues ω of the present problem given by

$$\omega^2 - \omega_0\omega - \left(\frac{mV_\theta}{r} + nV_z\right)^2 - \frac{V_\theta}{r}\frac{\omega_0}{\Omega}\left(\frac{mV_\theta}{r} + nV_z\right) = 0, \tag{6.2}$$

which has roots

$$\omega = \tfrac{1}{2}\omega_0 \pm \left\{\tfrac{1}{4}\omega_0^2 + \left(\frac{mV_\theta}{r} + nV_z\right)^2 + \frac{V_\theta}{r}\frac{\omega_0}{\Omega}\left(\frac{mV_\theta}{r} + nV_z\right)\right\}^{\frac{1}{2}} \tag{6.3}$$

$$= \tfrac{1}{2}\omega_0 \pm \left\{\left[\tfrac{1}{2}\omega_0 + \frac{V_\theta}{\Omega r}\left(\frac{mV_\theta}{r} + nV_z\right)\right]^2 + \left(\frac{mV_\theta}{r} + nV_z\right)^2\left(1 - \frac{V_\theta^2}{\Omega^2 r^2}\right)\right\}^{\frac{1}{2}}. \tag{6.4}$$

We first investigate the case $V_z = 0$, anticipating that since the left-hand side of (5.5) is zero for the equilibrium state assumed here any non-axisymmetric unstable modes must either have $|m| = 1$ or be of other than 'slow' wave type. In fact it turns out that they have *both* these properties, for when $V_z = 0$

$$\omega = \tfrac{1}{2}\omega_0 \pm \left\{\tfrac{1}{4}\omega_0^2 + \frac{m^2V_\theta^2}{r^2}\left(1 + \frac{\omega_0}{m\Omega}\right)\right\}^{\frac{1}{2}}. \tag{6.5}$$

The non-hydromagnetic eigenvalues ω_0 have the property $|\omega_0/\Omega| \leqslant 2$ (Chandrasekhar 1961). It therefore follows from (6.5) that only the $|m| = 1$ modes can be unstable and then (as evinced by (6.4)) only when $V_\theta^2 > \Omega^2 r^2$, so that they cannot possibly be of 'slow' wave type. Note also that $\omega_0 m\Omega$ must be negative if ω is to have a non-zero imaginary part, and that $\omega_R m\Omega$ is then also negative, so that the amplifying hydromagnetic waves propagate westward in accord with the general results of §4.

When the magnetic field has both azimuthal and axial components we note from (6.3) that instability implies that

$$\frac{V_\theta}{r}\frac{\omega_0}{\Omega}\left(\frac{mV_\theta}{r} + nV_z\right) < 0, \tag{6.6}$$

so that amplifying modes such that $|nV_z| < |mV_\theta/r|$ must propagate westward in accord with §4. Again $V_\theta^2 > \Omega^2 r^2$ is a necessary condition for their amplification, so that they cannot be of 'slow' wave type. It is natural to inquire whether this westward drift persists *whatever* the relationship between the axial and azimuthal components of magnetic field. In fact it does not, as we now show. Set, for example, $V_\theta/r = 4\Omega$ and $m = 1$. We are at liberty to choose $\omega_0 = \frac{1}{2}\Omega$, but this will then only be a legitimate eigenvalue of the non-hydromagnetic problem for certain n (see Chandrasekhar 1961). Whatever the value of n, now choose V_z so that $nV_z = -5\Omega$. This violates the condition $|nV_z| < |mV_\theta/r|$ by which amplifying waves are constrained to propagate westward, and the resulting eigenfrequencies obtained from (6.3) are then $\omega = \frac{1}{4}\Omega(1 \pm i\sqrt{15})$, one of which does indeed correspond to an *eastward*-propagating amplifying hydromagnetic wave.

Narrow gap between the cylinders

Suppose now that the radii of the cylinders are nearly equal so that the gap width $d = r_2 - r_1$ is very much less than the mean radius $r_M = \frac{1}{2}(r_1 + r_2)$. Suppose also that the magnetic field varies by a factor of order unity over a radial distance of order r_M (so that it varies by only a small amount across the gap between the cylinders). We shall focus attention in this example on modes for which the azimuthal wavelength is of order r_M while both radial and axial wavelengths are of order d. Thus m, nd, $d\hat{p}'/\hat{p}$ and $d^2\hat{p}''/\hat{p}$ will all be $O(1)$. We assume that both Q/F and G/F are $O(1)$, and justify this assumption *a posteriori*. All the terms in the coefficient of $\hat{p}$ in (2.16*a*) except $-Gn^2/F$ (which is of order d^{-2}) are then of order r_M^{-2}. The second term in the equation is clearly $O(\hat{p}/dr_M)$ and the first $O(\hat{p}/d^2)$. Thus we find to a first approximation (or, more precisely, with error $O(d/r_M)$)

$$\hat{p}'' - (Gn^2/F)\,\hat{p} = 0. \tag{6.7}$$

To the same approximation we may replace the function G/F in (6.7) by its value at the mean radius $r = r_M$. The equation then has solutions

$$\hat{p} \propto \cos l(r - r_1),$$

where the radial wavenumber l is an integral multiple of π/d in order that the boundary conditions (2.16*b*) be satisfied (to the appropriate order of accuracy). Thus $l^2F + n^2G = 0$, and on making the final assumption $(mV_\theta r^{-1} + nV_z)^2 \gg |\omega|^2$ (thus seeking 'slow' hydromagnetic wave solutions), again to be justified *a posteriori*, we obtain the following dispersion relationship:

$$\frac{2\Omega\omega}{mV_\theta/r + nV_z} \doteqdot -\frac{2V_\theta}{r} \pm \left\{\left(1 + \frac{l^2}{n^2}\right)\left(\frac{mV_\theta}{r} + nV_z\right)^2 - r\left(\frac{V_\theta^2}{r^2}\right)' + N^2\right\}^{\frac{1}{2}}. \tag{6.8}$$

Here all apparent 'variables' are to be regarded as evaluated at $r = r_M$.

We begin by considering the case when the magnetic field is purely azimuthal, i.e. $V_z = 0$. If $\mathscr{B} \equiv |N|^2 r_M^2/V_\theta^2$ does not greatly exceed unity $|\omega|$ will clearly be of order $V_\theta^2/\Omega r_M^2$. The various assumptions $Q/F \sim 1$, $G/F \sim 1$ and $m^2V_\theta^2/r^2 \gg |\omega|^2$ are then all valid provided only that $\Omega^2 r_M^2 \gg V_\theta^2$. With regard to the unstable modes given by (6.8) we note first that they all propagate westward in accord with the general result of §4. They may be caused by a radial increase of V_θ^2/r^2, a 'top-heavy' density gradient ($N^2 < 0$) or a combination of these effects. No unstable 'slow' modes occur unless

$$r(V_\theta^2/r^2)' - N^2 > m^2 V_\theta^2/r^2, \tag{6.9}$$

which is a slightly more stringent requirement than (5.5) (bearing in mind that $m^2/n^2r_M^2 \sim d^2/r_M^2 \ll 1$ in this example). Equation (6.9) displays the dual role played by the magnetic field: while its direct effect is to impart 'elasticity' to the system through the restoring force resulting from twisting of the 'equivalent elastic strings' (corresponding to the term $m^2V_\theta^2/r^2$) and hence to promote stability (as observed by Braginsky (1967), who briefly studies a configuration almost equivalent to this 'narrow-gap' system in the course of his investigation of the hydromagnetic convective instability of a rotating fluid sphere; see §7), this may be more than offset by the tendency for a radial increase of V_θ^2/r^2 to promote instability. We note in passing that (6.8) is evidently compatible with the bounds (4.9); it is only a shade more difficult to show that it is also compatible with the bounds (5.8) and (5.9).

The stabilizing influence of a 'bottom-heavy' density gradient ($N^2 > 0$) is evident from (6.8) and (6.9). Note (although the assumptions made in deriving (6.8) then require re-evaluation) that as $\mathscr{B}$ increases not only is the stabilizing effect stronger but the period of the wave is ultimately substantially decreased, so that when $\mathscr{B} \gg 1$ we have the curiously hybrid (non-amplifying) wave with frequency

$$\omega \doteqdot \frac{Nm}{2\Omega}\left(\frac{V_\theta}{r}\right)_{r=r_M} \tag{6.10}$$

(cf. Acheson & Hide 1973, equation (6.18)).

The eigenfrequencies of the *axisymmetric* modes may be calculated in a similar fashion. With $m = 0$ and rV_θ'/V_θ of order unity (as assumed previously) we again obtain $l^2F + n^2G = 0$. In this case, however, $F = -\omega^2$ and

$$G = -\omega^2 - r(V_\theta^2/r^2)' + N^2 + 4\Omega^2$$

(see equations (2.12)–(2.14)). Accordingly

$$\omega^2 = \frac{[4\Omega^2 + N^2 - r(V_\theta^2/r^2)']_{r=r_M}}{1 + l^2/n^2}, \tag{6.11}$$

and disturbances amplify only when (3.5) is satisfied, while

$$[r(V_\theta^2/r^2)' - N^2]_{r=r_M} > O[m^2V_\theta^2/r^2]_{r=r_M} \tag{6.12}$$

is sufficient for *non*-axisymmetric instability, as evinced by (6.8). As in the homogeneous case (Acheson 1972) these criteria differ widely at rapid rotation speeds; indeed (6.12) is remarkably similar to the narrow-gap criterion for the instability of the corresponding *non-rotating* system!† The reason is clear from

† This does not imply that the stability of the system cannot be guaranteed by sufficiently rapid rotation; one presumes that it *can*, but only inasmuch as both the frequencies and the growth rates of the amplifying slow modes steadily decrease as Ω increases (see (4.9), (5.9) and (6.8)) and *dissipative effects*, however small, ultimately suppress the instability mechanisms of this paper for some sufficiently large Ω; see Acheson & Hide 1973.

the arguments of §3: an exchange of two rings in a non-axisymmetric manner entails only a comparatively modest amount of work in order to twist the lines of force permeating the rings as they themselves get distorted in transit, and this is evidently what the right-hand side of (6.12) represents, being proportional to both the square of the magnetic field B_θ and the square of the amount of twisting m.

We have so far explored (6.8) only in the case $V_z = 0$. If, however, a small axial magnetic field component is present such that $V_z/V_\theta \sim d/r_M$ (thus ensuring, in view of the supposition here that $m \sim 1$ and $n \sim d^{-1}$, that $mV_\theta/r \sim nV_z$) the various approximations made in deriving (6.8) are still all covered by the requirement $\Omega^2 r_M^2 \gg V_*^2$. Evidently not all the amplifying waves are constrained to propagate westward, although this property still applies for those such that $|nV_z| < |mV_\theta/r|$, in keeping with the results of §4. Note also that *axisymmetric* slow modes are now possible. Further, the criterion for their amplification is comparable with that for their non-axisymmetric counterparts and has a similar physical interpretation. This disappearance of the otherwise crucial difference between stability with respect to (*a*) axisymmetric and (*b*) non-axisymmetric disturbances is due to the fact that when an axial magnetic field component is present the lines of force are distorted (and the constraint of angular momentum conservation accordingly relaxed) in both cases.

7. Concluding remarks

The analyses of §§5 and 6 concerning the circumstances in which non-axisymmetric instability occurs have been restricted to the case of uniform rotation. The problem when $(U_\theta/r)' \neq 0$ is apparently more difficult, but important to any serious geophysical application of this study (see below). If $U_\theta/r = \Omega + U_D/r$, where the velocity deviations U_D from 'rapid' rigid-body rotation are of order $V_* \Omega_*^{-1}(V_* r_*^{-1} + N_*)$ or smaller (so that $\omega/m = \sigma/m - U_\theta/r$ retains the typical 'slow' values assigned to it above *throughout the interval* $r_1 \leqslant r \leqslant r_2$, which it may not do otherwise), it is tempting to speculate here (by inspection of (2.15) and consideration of the modifications to (6.8) due to non-uniform rotation; see also Roberts & Soward 1972; Braginsky 1967) that the rough criterion (6.12) for 'slow' wave amplification is simply modified in the following way:

$$r\frac{d}{dr}\left(\frac{V_\theta^2}{r^2}\right) - 2\Omega r\frac{d}{dr}\left(\frac{U_D}{r}\right) - N^2 > O\left(\frac{m^2 V_\theta^2}{r^2}\right), \tag{7.1}$$

but the significance of this has yet to be demonstrated. Note that the new term would not in general be negligible unless U_D were small compared with a typical 'slow' wave speed.

With regard to the possibility expressed in A that the phenomena investigated there and in this paper may not be especially sensitive to the shape of the container (the earth's core, of course, being spherically bounded) we can offer at present only the following remarks. First, it would be compatible with the idea of hydromagnetic effects significantly relaxing the gyroscopic constraints due to rapid rotation on relative fluid motions (which is suggested by a variety of novel hydromagnetic phenomena in rotating fluids; see Acheson & Hide 1973), inasmuch as these strong constraints (most vividly exemplified, albeit in an extreme case, by 'Taylor columns'; see Greenspan 1968, p. 9) are essentially responsible for the extreme sensitivity of rapidly rotating *non*-hydromagnetic flows to the boundary conditions. Second, if (7.1) were to be of some relevance to spherical systems (notwithstanding the fact that the basic state would then typically be characterized by variations with both r *and* z) it would predict no amplifying '*slow*' modes in the special case $N = 0$, $U_D = V_\theta^2/2\Omega r$, which is in agreement with the results of an unpublished study by Booker (1972). Third, if the magnetic field is purely azimuthal and V_θ/r is constant the relation (6.5) between the eigenvalues ω and those (ω_0) of the corresponding *non*-hydromagnetic problem holds in a *sphere* (Malkus 1967), and since $|\omega_0/\Omega| \leqslant 2$ in that case also (Greenspan 1968, p. 52) there again only the $|m| = 1$ modes can be unstable (and then only if $V_\theta^2 > \Omega^2 r^2$) and, more significantly, they must propagate westward, just as in the cylindrical case.

It is unfortunate that in spite of the various developments outlined in this paper nothing approaching a simple physical picture of *why* the amplifying waves tend to propagate towards the west has yet emerged. Perhaps, in any case, it is misleading to emphasize this property in connexion with the geophysical problem at the expense of others. Thus, Braginsky (1964, 1967) has pointed out that by virtue of their asymmetry the waves are not encompassed by Cowling's anti-dynamo theorem, and the concomitant modifications to the mean state resulting from their amplification may form an integral part of the process by which the magnetic field is actually maintained against ohmic decay. Whichever aspect of the geophysical problem one has in mind their finite amplitude development is evidently of crucial importance (see Roberts & Soward 1972). Perhaps in this connexion we may conclude here with one elementary speculation. If (7.1) is indeed germane to the stability properties of the earth's liquid core it is clear that a radial increase of magnetic field has a destabilizing influence while a radial increase of angular velocity is stabilizing. These two will not be independent, however, for the azimuthal magnetic field will presumably be to some extent a consequence of the differential rotation 'winding up' the (somewhat smaller) meridional field. Accordingly, however the growth of the waves modifies the differential rotation (which will then have a greater or lesser stabilizing or destabilizing tendency, as the case may be), it seems likely that this will in turn change the azimuthal magnetic field in such a way that *it* then has precisely the *opposite* tendency as regards influencing the stability of the system and hence its future development, suggesting that the combination of these effects may constitute to some extent a self-regulation mechanism for the system as a whole.

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Appendix. Note on some effects of an axial shear flow

We consider here the *axisymmetric* stability of the system discussed in §3 when an axial shear flow $U_z(r)$ is also present. It then no longer follows that the eigenvalues are either real or purely imaginary. Making the transformation $\psi = \omega^{-\frac{1}{2}}\eta$ in (2.15), where $\omega = \sigma - nU_z$ (see equation (2.9)), multiplying by the complex conjugate of η and integrating between the boundaries we find

$$\int_{r_1}^{r_2} \omega r|\eta'|^2\,dr + \int_{r_1}^{r_2}\left(\frac{H}{\omega} + \frac{n^2U_zU_z'}{2\omega} - \frac{nU_z''}{2} - \frac{nU_z'}{2r}\right) r|\eta|^2\,dr = 0. \tag{A 1}$$

The imaginary part of this equation gives

$$\int_{r_1}^{r_2} \omega_I r|\eta'|^2\,dr + \int_{r_1}^{r_2}\{(H\omega^*)_I - \tfrac{1}{4}n^2(U_z')^2\,\omega_I\}\,|\omega|^{-2} r|\eta|^2\,dr = 0, \tag{A 2}$$

where ω^* denotes the complex conjugate of ω, and from this we conclude that any unstable modes must be such that $\omega_I^{-1}(H\omega^*)_I - \frac{1}{4}n^2(U_z')^2 < 0$ somewhere in the fluid. Using (2.15*b*) we find that this implies the following simple bound on the growth rates:

$$\omega_I^2 \leqslant \max\left\{r\frac{d}{dr}\left(\frac{V_\theta^2}{r^2}\right) - \frac{1}{r^3}\frac{d}{dr}(rU_\theta)^2 - N^2 + \frac{1}{4}\left(\frac{dU_z}{dr}\right)^2\right\}, \tag{A 3}$$

which in turn implies that a sufficient condition for axisymmetric stability is that the term in curly brackets is everywhere negative (cf. the *necessary and sufficient* condition of §3). The inequality (A 3), which is an elementary generalization of a result obtained by Howard & Gupta (1962), suggests that any axial shear flow, whatever its profile, tends to destabilize the system.

The present author shares their difficulty in obtaining any general results pertaining to the *non*-axisymmetric stability of the system. Even in the absence of hydromagnetic effects the problem is particularly interesting and challenging in view of the evidence from studies by Ludwieg (1962) and Pedley (1968) (see also Joseph & Munson 1970) that, despite the fact (as (A 3) clearly shows, if we set $V_\theta = N = 0$) that for the amplification of axisymmetric disturbances in a uniformly rotating fluid an axial shear flow such that $\frac{1}{4}(dU_z/dr)^2 > 4\Omega^2$ is necessary, a much smaller axial shear flow may suffice to destabilize the system in a *non*-axisymmetric manner (cf. the criteria (3.5) and (6.9)). Whether or not this link with the results of the present paper can be exploited to the mutual benefit of both hydromagnetic and non-hydromagnetic investigations remains to be seen.

Note added in proof. (i) Perhaps it should be made more explicit that all the results of this paper are derived on the understanding that n is non-zero. Unless this limitation of the analysis is borne in mind it is tempting to make inferences from it, particularly with regard to some related non-hydromagnetic problems, which are demonstrably incorrect. Suppose, for example, that the fluid is rotating uniformly. It may easily be shown from (2.15) that in the absence of a density gradient, or in the presence of a stable one, i.e. $N^2 \geqslant 0$, any modes amplifying as a result of a radial increase of magnetic field must then have $n \neq 0$. Modes generated by an unstable *density* gradient, on the other hand, need not be of this type. Further, while an immediate consequence of setting $n = 0$ in (2.15) would be $\sigma_R \sigma_I = 0$ (again assuming uniform rotation), i.e. no amplifying *wavelike* disturbances, this would not be the case in a bounded system whose depth changes with distance from the rotation axis. As an example we note the study by Busse (1970) of thermal instabilities in a rapidly rotating fluid sphere. Convection occurs in that case in the form of two-dimensional cells aligned with the rotation axis, and no information about such ($n = 0$) modes can legitimately be derived from the results of this paper. As Busse shows, they in fact propagate *eastward* in that case by essentially the classic Rossby wave mechanism.

(ii) More recent calculations by the author show that this supposition, while conceptually convenient, is not strictly necessary. Inequalities (5.7), (5.8) and (5.9) can be derived in roughly the manner shown merely by supposing that $|\omega|^2$ may be neglected in comparison with $m^2 V_\theta^2/r^2$ (the 'slow wave' approximation) without further reference to the relative sizes of the various wavelength components.

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